Root Finding For NonLinear Equations
Bisection Method

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Introduction

The growth of a population can be modeled over short periods of time by assuming that the population grows continuously with time at a rate proportional to the number present at that time.

If we let $N(t)$ denote the number at time t and λ denote the constant birth rate of the population, then the population satisfies the differential equation

$$\frac{dN(t)}{dt} = \lambda N(t).$$

The solution to this equation is

$$N(t) = N_0 e^{\lambda t},$$

where N_0 denotes the initial population.

This exponential model is valid only when the population is isolated, with no immigration. If immigration is permitted at a constant rate ν, then the differential equation becomes

$$\frac{dN(t)}{dt} = \lambda N(t) + \nu,$$

whose solution is

$$N(t) = N_0 e^{\lambda t} + \frac{\nu}{\lambda}(e^{\lambda t} - 1).$$

Suppose a certain population contains, $1,000,000$ individuals initially, that $435,000$ individuals immigrate into the community in the first year, and that $1,564,000$ individuals are present at the end of one year.

To determine the birth rate of this population, we must solve for λ in the equation

$$1,564,000 = 1,000,000e^{\lambda} + \frac{435,000}{\lambda}(e^{\lambda} - 1).$$

Numerical methods are used to approximate solutions of equations of this type, when the exact solutions cannot be obtained by algebraic methods.

Babylon and the Square Root of 2

We consider first one of the most basic problems of numerical approximation, the root-finding problem. This process involves finding a root, or solution, of an equation of the form $f(x) = 0$, for a given function f. A root of this equation is also called a zero of the function f.

The problem of finding an approximation to the root of an equation can be traced back at least as far as 1700 BC. A cuneiform table in the Yale Babylonian Collection dating from that period gives a sexigesimal (base-60) number equivalent to 1.414222 as an approximation to $\sqrt{2}$, a result that is accurate to within 10^{-5}.

Impressive Good Approximation to Square Root of 2

This is a Babylonian clay tablet from around 1700 BC. Its known as *YBC*7289, since its one of many in the Yale Babylonian Collection. Its a diagram of a square with one side marked as having length 1/2.

They took this length, multiplied it by the square root of 2, and got the length of the diagonal. Since the Babylonians used base 60, they thought of 1/2 as 30/60. But since they hadnt invented anything like a decimal point, they wrote it as 30. More precisely, they wrote it as this:

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.41421297....$$

This is an impressively good approximation to $\sqrt{2} \approx 1.41421356....$

Order of Convergence

A sequence of iterates $\{x_n : n \geq 0\}$ is said to converge with order $p \geq 1$ to a point α if

$$|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p, \qquad n \geq 0 \tag{1}$$

for some $c > 0$, called asymptotic error constant.

If $p = 1$, the sequence is said to converge linearly to α. In that case, we require $c < 1$; the constant c is called the *rate of linear convergence* of x_n to α.

In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order.

The asymptotic constant affects the speed of convergence but is not as important as the order.

Bisection Method

The **bisection method** is based on the Intermediate Value Theorem.

Suppose f is a continuous function defined on $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. By the Intermediate Value Theorem, there exists a number α in (a, b) with $f(\alpha) = 0$.

Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique. The method calls for a repeated halving of subintervals of $[a, b]$ and, at each step, locating the half containing α.

Algorithm

To begin, set $a_1 = a$ and $b_1 = b$, and let x_1 be the midpoint of $[a, b]$. That is,

$$x_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2} \quad \texttt{(first approximation).}$$

If $f(x_1) = 0$, then $\alpha = x_1$, and we are done. If $f(x_1) \neq 0$, then $f(x_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.

1. When $f(a_1)$ and $f(x_1)$ have the same sign, $\alpha \in (x_1, b_1)$, and we set $a_2 = x_1$ and $b_2 = b_1$.
2. When $f(a_1)$ and $f(x_1)$ have opposite signs, $\alpha \in (a_1, x_1)$, and we set $a_2 = a_1$ and $b_2 = x_1$.

We then reapply the process to the interval $[a_2, b_2]$ to get `second approximation` p_2.

Stopping Procedures

We can select a tolerance $\varepsilon > 0$ and generate $p_1, p_2, \ldots, p_N$ until one of the following conditions is met.

$$\begin{aligned} |p_N - p_{N-1}| &< \varepsilon, && (2) \\ |f(p_N)| &< \varepsilon, && (3) \\ \frac{|p_N - p_{N-1}|}{|p_N|} &< \varepsilon, \quad p_N \neq 0. && (4) \end{aligned}$$

Difficulties can arise when the first and second stopping criteria are used. For example,

- sequences $(p_n)_{n=1}^{\infty}$ with the property that the differences $p_n - p_{n-1}$ can converge to zero while the sequence itself diverges.
- It is also possible for $f(p_n)$ to be close to zero while p_n differs significantly from α.

How to apply bisection algorithm?

- An interval $[a, b]$ must be found with $f(a).f(b) < 0$.
- As at each step the length of the interval known to contain a zero of f is reduced by a factor of 2, it is advantageous to choose the initial interval $[a, b]$ as small as possible. For, example, if $f(x) = 2x^3 - x^2 + x - 1$, we have both

$$f(-4).f(4) < 0 \text{ and } f(0).f(1) < 0,$$

so the bisection algorithm could be used on either on the intervals $[-4, 4]$ or $[0, 1]$. However, starting the bisection algorithm on $[0, 1]$ instead of $[-4, 4]$ will reduce by 3 the number of iterations required to achieve a specified accuracy.

Best Stopping Criterion

Without additional knowledge about f and α, inequality

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, p_N \neq 0,$$

is the best stopping criterion to apply because it comes closest to testing relative error.

For example, the iteration process is terminated when the relative error is less then 0.0001; that is, when

$$\frac{|\alpha - p_n|}{|\alpha|} < 10^{-4}.$$

How to find the relative error bound?

How to find $\frac{|\alpha - p_n|}{|\alpha|}$, relative bound? The following theorem answers the question.

Theorem

Suppose that $f \in C[a,b]$ and $f(a).f(b) < 0$. The bisection method generates a sequence $(p_n)_{n=1}^{\infty}$ approimating a zero α of f with

$$|p_n - \alpha| \leq \frac{b-a}{2^n}, \text{ when } n \geq 1.$$

- The method has the important property that **it always converges to a solution**, and for that reason it is often used as a starter for the more efficient methods. But it is slow to converge.
- It is important to realize that the above theorem gives only a bound for approximation error and that this bound may be quite conservative.

Number of Iterations Needed?

To determine the number of iterations necessary to solve

$$f(x) = x^3 + 4x^2 - 10 = 0$$

with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$ requires finding an integer N that satisfies

$$|p_N - \alpha| \leq 2^{-N}(b-a) = 2^{-N} < 10^{-3}.$$

A simple calculation shows that ten iterations will ensure an approximation accurate to within 10^{-3}.

Again, it is important to keep in mind that the error analysis gives only a bound for the number of iterations, and in many cases this bound is much larger than the actual number required.

How to find an approximation which is correct to at least some 's' significant digits?

Find the iteration number n_0 such that $\frac{|b_{n_0} - a_{n_0}|}{|a_{n_0}|} \leq \frac{1}{10^{s+1}}$.

Since

$$\frac{|\alpha - p_{n_0-1}|}{|\alpha|} \leq \frac{|b_{n_0} - a_{n_0}|}{|a_{n_0}|} \leq \frac{1}{10^{s+1}},$$

p_{n_0-1} is an approximation which is correct to at least some 's' significant digits.

As you have calculated p_{n_0}, so you can take p_{n_0}, which is a (good) approximation which is correct to at least some 's' significant digits.

Order of Convergence

Let p_n denote the nth approximation of α in the algorithm. Then it is easy to see that

$$\alpha = \lim_{n\to\infty} p_n$$

$$|\alpha - p_n| \le \left[\frac{1}{2}\right]^n (b-a) \tag{5}$$

where $b - a$ denotes the length of the original interval. From the inequality $|\alpha - p_n| \le \frac{1}{2}|\alpha - p_{n-1}| \quad n \ge 0$, we say that the bisection method converges linearly with a rate of $\frac{1}{2}$.

The actual error may not decrease by a factor of $\frac{1}{2}$ at each step, but the average rate of decrease is $\frac{1}{2}$.

References

- Richard L. Burden and J. Douglas Faires, "*Numerical Analysis – Theory ad Applications*", Cengage Learning, New Delhi, 2005.
- Kendall E. Atkinson, "*An Introduction to Numerical Analysis*", John Wiley & Sons, Delhi, 1989.